Received: *August 28, 2000*, Accepted: *January 24, 2001*
Revised: *January 24, 2001*

Finite temperature effects in the supergravity dual of the $N = 1^*$ gauge theory

Daniel Z. Freedman

Department of Mathematics and Center for Theoretical Physics
Massachusetts Institute of Technology, Cambridge
MA 02139, USA
E-mail: dzf@math.mit.edu, dzf@bourbaki.mit.edu

Joseph A. Minahan*

Center for Theoretical Physics, Massachusetts Institute
of Technology, Cambridge, MA 02139, USA
E-mail: minahan@mit.edu

Abstract: We consider the supergravity dual of the $\mathcal{N} = 1^*$ theory at finite temperature by applying the Polchinski-Strassler construction to the black D3-brane solution of type-IIB supergravity. At finite temperature the 5-brane probe action is minimized when the probe falls to the horizon, although metastable minima with $r \gg r_H$ persist for a range of temperatures. Thermal effects on the 3-form source for the chiral multiplet mass m and its order m^2 back reaction on the other fields of the IIB theory are computed. We find unique solutions which are regular at the horizon and have the correct behavior on the boundary. For fixed temperature T, the horizon shrinks for increasing m^2 suggesting that there is a critical temperature separating the system into high and low temperature phases. In the high temperature phase 5-branes are unnecessary since there are no naked singularities. Using the order m^2 correction to the horizon area we calculate the correction to the entropy to be $\Delta\mathcal{S} = -0.1714N^2m^2T$, which is less than the free field result.

Keywords: Brane Dynamics in Gauge Theories, AdS/CFT Correspondance, Black Holes.

*Address after Sept. 1, 2000: Department of Theoretical Physics, Uppsala University, Uppsala, Sweden

JHEP01(2001)036

Contents

1. Introduction

The Maldacena conjecture [1] relating classical supergravity to strongly coupled conformal gauge theories has had many successes [2]. Perhaps more remarkable is the fact that one can apply it to non-conformal theories as well. The first, and perhaps most famous application has been to $\mathcal{N} = 4$ super Yang-Mills at finite temperature [3]. One may consider this as a gauge theory compactified on a circle where the fermions have anti-periodic boundary conditions, breaking the supersymmetry. This dimensionally reduced gauge theory is expected to be confining with a mass gap. By arguing that this theory has a supergravity dual which is a black hole in a curved space, Witten was able to demonstrate both the existence of an area law and a mass gap in the strong coupling limit.

One would like to find other models that exhibit confining behavior. Such examples have been found, either by adding spin to the black hole, breaking some of the global symmetries of the gauge theories [4]–[8], or by considering more exotic theories such as type-0 string theories [9]–[12] or even type-II theories with a dilaton turned on [13, 14, 15], breaking the $\mathcal{N} = 4$ supersymmetry. In these latter cases, while there appear to be gaps and area laws, there are also naked singularities which would seem to destroy the viability of these theories. However, singularities are not necessarily disasters. Some of them may be hidden behind horizons [16]. Alternatively, they may be resolved by stringy considerations.

Recently, Polchinski and Strassler considered another model [17], the $\mathcal{N} = 1^*$ model, in which mass terms for the three chiral multiplets that make up an $\mathcal{N} = 4$ multiplet are included. Several interesting generalizations of [17] have also appeared [18]–[23]. At high energies, the theory is the usual $\mathcal{N} = 4$ model, while at low energies, the supersymmetry is broken to $\mathcal{N} = 1$. Unlike gauge theories with higher supersymmetries, $\mathcal{N} = 1$ theories can be confining. In fact, this gauge theory has a large class of degenerate but discrete vacua [24, 25] which include Higgs, confining, and oblique confining phases. In these cases there is a mass gap, but there are also vacua with unbroken U(1) subgroups and thus massless photons.

Polchinski and Strassler constructed the supergravity dual of this model as follows. They assumed that a 3-form field strength is turned on whose strength is proportional to the mass m of the chiral multiplets. The field strength induces a Myers dielectric effect on the D3-branes, essentially expanding them out in the transverse directions to a two sphere [26, 27]. The two sphere is effectively a D5-brane, or one of its S-duals, with n units of magnetic flux. These flux units minimally couple to the 4-form Ramond-Ramond gauge potential, and hence they correspond to D3-brane charges. The background metric, for large distances, that is for the UV limit, is dominated by the D3-brane charges. But as one moves toward the IR, one starts probing near the D5-brane and here, the metric is dominated by the D5-brane. In between the metric interpolates between these two limits.

The presence of this D5-brane significantly changes the physics [17]. For example, at large separations, magnetic charges are effectively confined, since a D string will have finite tension as it approaches the D5-brane. However, the electric charges are screened, since the fundamental string can end on the D5-brane. Likewise, for the S-dual picture, electric charges are confined since a fundamental string has finite tension near an NS5-brane, but the magnetic charges are screened because the D-string can end on the NS5-brane.

Polchinski and Strassler also demonstrated how the supergravity duals map into the various degenerate vacua of the $\mathcal{N} = 1^*$ theory. The D3-branes can expand into more than one 5-brane and in the regime where the supergravity limit *is valid*, one can find a one to one correspondence between the various ways of dividing up D3-branes into 5-branes and the degenerate vacua. For these vacua, one expects a gap, in other words, the absence of massless particles. Actually, this is not completely true, since many vacua have unbroken U(1) gauge groups. However, the number of massless states is at most of order $\sqrt{N}$ and their effects are not seen in supergravity duals.

If one were to consider the $\mathcal{N} = 1^*$ theory at finite temperature, then the multiple vacua and gaps have profound effects. Consider first the entropy. The entropy of a massless theory in four dimensions scales as T^3. However, a theory with a gap will have its entropy suppressed exponentially when the temperature falls below the mass gap scale. In the large-N limit, this should occur as a phase transition between high and low temperature phases, where at some critical temperature the free energies of the two phases is equalized. Another effect of finite temperature is that the degeneracy between the separate vacua should be lifted, leading to many metastable states. As the temperature is raised above the mass gap scale, these metastable vacua should disappear altogether, and at very high temperatures, the theory should behave just like the $\mathcal{N} = 4$ theory at finite temperature.

In the supergravity dual of these theories, the high temperature phase is described by a black hole in an asymptotic AdS space. At high temperatures there is a unique vacuum. The multiple vacuum configurations of 5-branes described by Polchinski and Strassler will now have a higher free energy than the vacuum. The 5-branes were introduced to account for naked singularities that inevitably appear and which were specifically discussed in [28]. However, at high temperatures, singularities are shielded by the horizon; hence, there is no need for the 5-branes.

For the high temperature phase, if one considers a spherical 5-brane probe with some D3-brane charge, then one should find that the free energy is minimized when the 5-brane lies on the horizon. One should also find that the energy is the same as if there were only D3-branes and no 5-branes, since the 5-brane has no net charge, and hence would be undetectable at the horizon.

However, in the high temperature phase, there still may exist local minima for the 5-branes outside the horizon. These configurations correspond to metastable vacua, valid in the probe limit described in [17], and which can be in a partial Higgs, confining or oblique confining phase. As the temperature is increased, these local minima eventually disappear.

A crucial step for going beyond the probe calculation is the realization that the probe calculation itself is unchanged for an arbitrary distribution of parallel D3-branes [17]. Hence, one can construct a spherical shell of D3-branes, of order N in number, and still use the probe calculation to determine the radius of the shell. This is not to say that the exact supergravity solution has been constructed, since one still needs to compute the back reaction. But it does allow one to build an interpolating metric between a D3-brane geometry and a 5-brane geometry. However, in the finite temperature case, one cannot use this approach as a starting point, since a configuration of D3-branes outside the horizon is not a solution to Einstein's equations.

One can also explore the effects of the chiral multiplet mass on the entropy in the high temperature phase. If the chiral multiplets are given a small mass m, then there are polynomial corrections to the entropy. In particular, to order m^2 the free field limit of the entropy is given by

$$\mathcal{S} = \frac{2\pi^2}{3} N^2 T^3 - \frac{3}{4} N^2 m^2 T + \mathrm{O}(m^4)\,. \tag{1.1}$$

The strong coupling calculation for the massless case was carried out in [29], where the area of a blackhole in AdS space was computed. These authors found that

$$\mathcal{S}_0 = \frac{\pi^2}{2} N^2 T^3 \,, \tag{1.2}$$

a factor of 3/4 from the free theory result. An immediate question is to find the massive corrections in the supergravity limit and compare this with the free field result.

In this paper we study the finite temperature effects outlined above for the supergravity dual of the $\mathcal{N} = 1^*$ theory. Our starting point is the black D3-brane solution [30] of type-IIB supergravity perturbed by a 3-form gauge potential. A 5-brane probe calculation quickly reveals several features of finite temperature. First, the $T = 0$ minima [17] for 5-branes wrapping 2-spheres of particular non-zero radii are now only local minima; the true minimum appears at the horizon, $r = r_H$ where the free energy is minimized. The local minima, which can be thought of as metastable vacua, move to smaller radii and eventually disappear as the temperature is increased, but at temperatures that can be much higher than when the field theory is weakly coupled. We demonstrate that at the horizon a wrapped 5-brane, which carries n units of D3-brane charge, cannot be distinguished from a configuration of n D3-branes.

Although the major thermal effects in the probe calculation can be obtained quite simply, other questions require the finite temperature modification of the fields of the type-IIB theory through order m^2. We therefore compute exactly the 3-form field strength in the black D3-brane background. The metric, 5-form field strength and dilaton are then sourced at order m^2 by bilinears in the 3-form, and we solve the equations which determine the back reaction to this order. Given the modified metric, the relation between temperature T and horizon distance r_H is recomputed, and we find that the horizon shrinks for fixed T by an amount of order $\Delta r_H \sim m^2 R^2/T$ where R is the AdS scale. We presume that a more complete calculation would show that $r_H \to 0$ at some temperature T_c which is then the critical temperature for the phase transition in the theory.

The computations for the back reaction can be specialized to the $T = 0$ case, where we must require that the homogeneous modes agree in their SO(6) representation and r-dependence at the boundary with fields of the dimensionally reduced IIB supergravity [31]. In general the order m^2 inhomogeneous solutions approach the boundary at different rates than the homogeneous modes. For example, the dilaton perturbation equation is that of the second Kaluza-Klein excitation, the $\Delta = 6$ field in the **20'** representation with an order m^2 source. The **20'** representation agrees with the general relation between the $D = 10$ and the $D = 5$ dilaton postulated in [33], but the solution does not agree with the order m^2 term in the interpolating dilaton of [17]. The back reaction equations of metric components transverse to the D3-branes describe scalars with $\Delta = 8$ in the singlet representation and $\Delta = 6$ in the

84 representation of SO(6). We also show that the back reaction on the metric and 5-form lead precisely to the order m^2 terms in the probe action, which were inserted in [17] as a requirement of supersymmetry. All of these $T = 0$ results are a useful check that the rather complicated back reaction equations are correct.

We also compute the decreased area of the horizon in the (approximate) modified geometry and thus obtain the entropy

$$\mathcal{S}_0 = \frac{\pi^2}{2} N^2 T^3 - 0.1714 N^2 m^2 T + \mathrm{O}(m^4) \tag{1.3}$$

including the order m^2 correction. Comparing with (1.1) we see that the large-N correction is 23% of the free field result.

Technically speaking, it is noteworthy that finite temperature effects can be computed essentially analytically as far as we have gone, and this raises the hope of going beyond the leading order m^2 approximation. At $T = 0$ this proved to be easy [17] since the probe calculation is not altered when the D3-branes are expanded around 2-spheres. This fortuitous simplification is unlikely at finite T, since the only solution to the equations of motion when $m = 0$ has all D3-branes at the origin.

In section 2 we consider the probe calculation. In section 3 we compute the linear solutions for the 3-form field strengths in a finite temperature D3 background. In section 4 we obtain the back reaction to the dilaton field. Order m^2 temperature dependent corrections to the metric and 5-form are discussed in section 5, and the entropy calculation is presented in section 6. The appendix A contains a more detailed calculation for the disappearance of metastable vacua.

2. Probe calculation for the vacua

In this section[1] we look for non-trivial solutions for a D5-brane probe in a finite temperature D3-brane background geometry. Temperature is introduced via the thermal D3-brane solution [30] of type-IIB supergravity. The metric is

$$\begin{aligned} ds^2 &= (Z(r))^{-1/2}\left[-f(r)dt^2 + d\mathbf{x}^2\right] + Z(r)^{1/2}\left[\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right], \\ Z(r) &= \frac{R^4}{r^4}, \\ f(r) &= 1 - \frac{r_H^4}{r^4}, \end{aligned} \tag{2.1}$$

where $R^4 = 4\pi g N \alpha'^2$. The solution has a horizon at $r = r_H$, which corresponds to the temperature $T = r_H/(\pi R^2)$. The accompanying dilaton, axion and self-dual

[1] the notation of [17] is used throughout the present paper

5-form are given by [17]

$$e^{\Phi} = g\,, \tag{2.2}$$
$$C = \frac{\theta}{2\pi}\,, \tag{2.3}$$
$$\tilde{F}_5 = d\chi_4 + *d\chi_4\,, \tag{2.4}$$
$$\chi_4 = \frac{dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3}{gZ(r)}\,. \tag{2.5}$$

While thermal effects cancel out for the $*$ operation contained in $\tilde{F}_5$, they will play an important role when $*$ acts on other field strengths.

The action for the D5-brane is the sum of Born-Infeld and Chern-Simons parts, and is given by

$$\begin{aligned} S = -\frac{\mu_5}{g}\int d^6\xi \left[-\det(G_{\|})\det\left(g^{-1/2}e^{\Phi/2}G_{\perp} + 2\pi\alpha'\mathcal{F}\right)\right]^{1/2} + \\ + \mu_5 \int \left(C_6 + 2\pi\alpha'\mathcal{F}_2 \wedge C_4\right), \end{aligned} \tag{2.6}$$

where

$$2\pi\alpha'\mathcal{F}_2 = 2\pi\alpha' F_2 - B_2\,. \tag{2.7}$$

As in [17], $G_{\|}$ refers to the pullback of the metric along the R^4, while $G_{\perp}$ is the pullback of the metric onto the S^2 on which the 5-brane is wrapped. C_4 is the Ramond-Ramond gauge potential of the D3-branes, while C_6 and B_2 are potentials for the 3-form perturbation of the background dual to the mass operator $m\,\mathrm{tr}(\Phi_1^2 + \Phi_2^2 + \Phi_3^2)$ of the $\mathcal{N} = 1^*$ field theory.

The field F_2 is the U(1) gauge field strength of the wrapped 5-brane, and is assumed to contain n units of flux, so that

$$\int_{S^2} F_2 = 2\pi n\,. \tag{2.8}$$

Since F_2 couples to C_4 in the Chern-Simons term of (2.6), n is the effective D3 charge of the 5-brane. As in [17], we assume that the first term in (2.6) is dominated by F_2. The condition $n^2 \gg gN$ must then hold in order for the probe calculation to be valid [17].

In the zero temperature limit, the leading order $n/\sqrt{gN}$ term from the Born-Infeld part of (2.6) cancels with the C_4 term in the Chern-Simons part. The radius of the S^2 on which the D5-brane resides is then chosen to minimize the sum of first order correction terms. At finite temperature, leading terms no longer cancel because $\det G_{\|}$ is modified by the Schwarzschild factor [32], viz.

$$\det(G_{\|}) = Z^{-2}\left(1 - \frac{r_H^4}{r^4}\right). \tag{2.9}$$

There is then a residual r-dependent term which must be added to the non-leading terms computed in [17]. The sum of all these terms is then minimized to find the favored position of the probe. The modification of $\det G_{||}$ is the major thermal effect on the probe calculation. Thermal modification of other quantities is considered in later sections of this paper and produces corrections of higher order in r_H^2/r^2 which can be ignored when the 5-brane is far from the horizon. The near-horizon effect of these corrections will be included qualitatively here and then supported by the work of later sections. We now proceed to implement the probe calculation just outlined.

At finite temperature, the contribution to the action from the Born-Infeld term in (2.6) which is of leading order in the expansion parameter $n/\sqrt{gN}$ is

$$\begin{aligned} S_0 &= -\frac{2\pi\alpha'\mu_5}{g}\int d^6\xi \left(-\det(G_{||})\right)^{1/2}\left(\det(F_2)\right)^{1/2} \\ &= -\frac{\mu_5}{gn}4\pi^2\alpha'\frac{n^2r^4}{R^4}V\sqrt{1-\frac{r_H^4}{r^4}}\,, \end{aligned} \tag{2.10}$$

where r_H is the position of the horizon and V is the volume of $\mathbf{R}^4$. The leading contribution from the Chern-Simons term is

$$2\pi\alpha'\mu_5\int \mathcal{F}_2\wedge C_4 = +\frac{\mu_5}{g}4\pi^2\alpha'\frac{nr^4}{R^4}V\,. \tag{2.11}$$

Dominant thermal corrections are absent since the metric does not appear in this term. Further only the parallel components of the Ramond-Ramond gauge potential contribute to the integral, so we have set $C_4 = \chi_4$. The zero temperature first order correction to the Born-Infeld term plus the contribution of the C_6 plus the order m^2 term required by supersymmetry is given in [17] as

$$\Delta S = -\frac{2\mu_5}{g\alpha' n}r^2(r-r_0)^2V\,, \tag{2.12}$$

where

$$r_0 = \pi|m|n\alpha'\,. \tag{2.13}$$

Adding the terms in (2.10), (2.11) and (2.12) leads to the expression

$$-\frac{S}{V} = \frac{2\mu_5}{g\alpha' n}\left(\frac{\pi n^2}{2gN}r^4\left(\sqrt{1-\frac{r_H^4}{r^4}}-1\right)+r^2(r-r_0)^2\right). \tag{2.14}$$

This result for the free energy of the probe is valid if $r \gg r_H$. Note that for $r_H = 0$ (2.14) has a minimum at $r = r_0$ and a maximum at $r = r_0/2$. We now discuss thermal effects in this region of r. For $r_H \ll r$ the minimum and maximum shift by

$$\begin{aligned} \Delta r_{\min} &= -\frac{\pi n^2 r_H^8}{2gNr_0^7}\,, \\ \Delta r_{\max} &= +\frac{32\pi n^2 r_H^8}{gNr_0^7}\,. \end{aligned} \tag{2.15}$$

Therefore, as r_H is increased, the maximum and minimum are pushed toward each other, and the minimum disappears when $\Delta r_{\rm max} \sim r_0$. This occurs at the critical horizon size $r_H \sim (gN/n^2)^{1/8} r_0$ which is well inside of r_0 and thus within the region of validity of the calculation.

Next let us compare the energies for a D5-brane at $r = R_H$ and at $r = r_0$. Using the expressions in (2.14) and (2.13) and the relation between R_H and T, we find that the probe brane has less energy at the horizon if

$$T > \pi^{-1} m \,. \tag{2.16}$$

Notice that the crossover point is independent of n. This suggests that a phase transition occurs at such a temperature.

However, we believe that the actual mechanism for the phase transition is somewhat different than this naive picture due to thermal corrections to the second term in the r.h.s. of (2.14). These corrections are important near the horizon. One effect is that the r^4 term in (2.12) should be multiplied by $\sqrt{1 - r_H^4/r^4}$ since it comes from the Born-Infeld part of (2.6). There are other effects from the 3-form perturbation and its back reaction on the metric and 5-form. The linear term in r_0 comes from the Chern-Simons term

$$\mu_5 \int (C_6 - B_2 \wedge C_4) \,, \tag{2.17}$$

and goes to zero at the horizon due to a conspiracy between C_6 and B_2. We should expect this because standard no-hair theorems imply that the D5-brane at the horizon is equivalent to n D3-branes, since there is no net D5-brane charge. At the horizon, there is no contribution from the Born-Infeld action. Hence, one would need to verify that the actions coming from the Chern-Simons pieces are equivalent. The difference between the two actions is proportional to (2.17). We will show that the order m^2 contribution is zero at the horizon.

Finally, and most importantly, the r_0^2 term has contributions from both the Born-Infeld and Chern-Simons parts of the probe action. The Born-Infeld contribution is again suppressed by the factor $\sqrt{1 - r_H^4/r^4}$, but the Chern-Simons term also vanishes since the order m^2 corrections to C_4 approach zero. These statements will be verified to order m^2 in sections 3 and 4.

Hence, at least to order m^2, the second term on the r.h.s. of (2.14) goes to zero as $r = r_H$. We suspect that this will be true to all orders in m. Therefore, it appears that the 5-brane has lower free energy at the horizon for any $r_H > 0$, so that the minimum near $r = r_0$ is only a metastable state and is a degenerate vacuum state only if $r_H = 0$. Therefore, the system is in a high temperature phase whenever $R_H > 0$.

How then does the phase transition occur? There is still another thermal effect which we discuss in section 6. Due to the back reaction of the 3-form perturbation on the metric the horizon size itself shrinks as a function of m^2 for fixed black hole temperature. Hence, we believe that r_H vanishes at some finite temperature T_0 of order $T_0 \sim m$. This value would then be a critical temperature for the probe calculation. In the low temperature phase $T < T_0$ the 5-brane is stabilized near $r = r_0$; in the high temperature phase $T > T_0$ it retreats to the horizon. Work is continuing to verify this suggested picture.

The generalization of the previous analysis to NS5 probe branes with p units of D3-brane charge is straightforward. To proceed, one describes the D3 background using the S-dual description

$$\tau' = \frac{a\tau + b}{c\tau + d} \tag{2.18}$$

with field transformations

$$\begin{aligned} g' = g|M|^2\,, \qquad G'_{MN} = G_{MN}|M|\,, \qquad C_4 = C_4\,, \\ G'_3 = G_3\,, \qquad B'_6 - \tau' C'_6 = (B_6 - \tau C_6)M^{-1}\,, \end{aligned} \tag{2.19}$$

where $M = c\tau + d$. Hence, the leading order term in (2.14) is invariant under this transformation. The first order terms have the same transformation as in [17],

$$\begin{aligned} \Delta S &= \frac{2\mu_5}{g^3\alpha' p} r^2 (r - \widetilde{r}_0)^2 V\,, \\ \widetilde{r}_0 &= \pi g|m|p\alpha'\,. \end{aligned} \tag{2.20}$$

Hence, the total action that is minimized is

$$-\frac{S}{V} = \frac{2\mu_5}{g^3\alpha' p}\left(\frac{\pi g p^2}{2N} r^4 \left(\sqrt{1 - \frac{r_H^4}{r^4}} - 1\right) + r^2(r - \widetilde{r}_0)^2\right). \tag{2.21}$$

Finally we point out that a more precise analysis in the appendix A shows that the metastable D5 probe solution near $r = r_0$ disappears above the temperature

$$\begin{aligned} T &\approx \kappa \left(\frac{n^2}{4\pi g N}\right)^{3/8} |m|\,, \\ \kappa &= \frac{1}{16}\left(\frac{54}{\pi^2}\right)^{1/8} \left(3351797 + 171(2659)\sqrt{57}\right)^{1/8} \approx 0.5522\,. \end{aligned} \tag{2.22}$$

Likewise, for NS5-brane probes one finds that the solution disappears at the temperature

$$T \approx \kappa \left(\frac{g p^2}{4\pi N}\right)^{3/8} |m|\,, \tag{2.23}$$

Hence these metastable probe minima can survive well above the critical temperature, at least for those probes that satisfy the probe condition $n^2 \gg gN$. This contrasts with the weak coupling analysis. At weak coupling, the presence of temperature introduces the effective mass term $T^2\,\mathrm{tr}(\phi^2)$ to the effective potential. Hence, there are metastable Higgs vacua here as well. However, in this case these local minima are washed out when $T \sim m$.

What is intriguing about these metastable vacua is that one could have solutions where only some of the D3-brane charge is outside the horizon. The corresponding vacuum is a metastable state where say part of the SU(N) gauge group has been Higgsed, but the rest is unbroken and unconfined. Likewise, we can have metastable vacua where part of the gauge group is confined, but the rest is unbroken and unconfined.

3. The 3-form perturbation

As in [17] the first step in the study of thermal effects is to obtain linear perturbations of H_3 and $\tilde{F}_3 = F_3 - CH_3$ which are dual to fermion mass terms in the boundary gauge theory. Specifically we must solve the linearized equations of motion and Bianchi identities in the background (2.1) and (2.2)–(2.4):

$$\begin{aligned} d * \tilde{F}_3 &= F_5 \wedge H_3 \,, \\ d * \left(\frac{1}{g} H_3 - gC\tilde{F}_3 \right) &= -gF_5 \wedge F_3 \,, \\ d\tilde{F}_3 &= 0 = dH_3 \,. \end{aligned} \tag{3.1}$$

Expressing the fields as the complex combinations

$$G_3 = F_3 - \tau H_3 \,, \tag{3.2}$$

$$\tau = C + \frac{i}{g} \,, \tag{3.3}$$

the equations of motion in (3.1) can be recast into the compact form [17]

$$\begin{aligned} d * G_3 + igG_3 \wedge F_5 &= 0 \,, \\ dG_3 &= 0 \,. \end{aligned} \tag{3.4}$$

The black hole metric affects only the radial dependence of G_3, the dependence on angles must be that of the lowest spherical harmonic on S^5 in order to be a source for the fermion mass in the **10** or $\overline{\mathbf{10}}$ representation of SO(6). Therefore we postulate that G_3 is the combination

$$G_3 = \alpha(r) T_3 + \beta(r) V_3 \tag{3.5}$$

of the same 3-forms used in [17]. The 3-forms T_3 and V_3 are constructed from a constant antisymmetric tensor T_{mnp} which is either self-dual or anti-self-dual corresponding to the **10** or $\overline{\mathbf{10}}$ representations, respectively. These forms are simply written in terms of Cartesian coordinates $y^m, m = 1, 2, \cdots, 6$ in the space perpendicular to the 3-branes, with $r^2 = y^m y^m$:

$$\begin{aligned} T_3 &= \frac{1}{3!} T_{mnp} dy^m \wedge dy^n \wedge dy^p \,, \\ S_2 &= \frac{1}{2} T_{mnp} y^m dy^n \wedge dy^p \,, \\ V_3 &= d(\ln r) \wedge S_2 \,. \end{aligned} \tag{3.6}$$

One can easily derive the properties

$$
\begin{aligned}
*_6 T_3 &= \pm i T_3\,, \\
*_6 V_3 &= \pm i(T_3 - V_3)\,, \\
dT_3 &= 0\,, \\
dS_2 &= 3T_3\,, \\
dV_3 &= -3d(\ln r) \wedge T_3\,,
\end{aligned} \tag{3.7}
$$

where $*_6$ is the Poincare duality operation in flat R^6.

We now proceed to solve eqs. (3.4). Substitution of the ansatz (3.5) in the Bianchi identity readily gives

$$
dG_3 = \left(\alpha'(r) - \frac{3\beta(r)}{r}\right) dr \wedge T_3 = 0 \tag{3.8}
$$

which determines the relation $\beta(r) = r\alpha'(r)/3$. One can then directly show that

$$
G_3 = \frac{1}{3} d(\alpha(r) S_2)\,. \tag{3.9}
$$

The equation of motion in (3.4) contains the duality operation with respect to the full metric of (2.1),

$$
*G_3 = \frac{\sqrt{-g}}{3!} \epsilon_{mnpqst} g^{qq'} g^{ss'} g^{tt'} G_{q's't'} dy^m \wedge dy^n \wedge dy^p \wedge gZ(r)\chi_4\,, \tag{3.10}
$$

where g_{mn} is the transverse piece of the metric of (2.1) rewritten in terms of y^m,

$$
g_{mn} = \left(\delta_{mn} - \frac{y^m y^n}{r^2}\right) + f^{-1}(r) \frac{y^m y^n}{r^2}\,. \tag{3.11}
$$

When compared to the case of zero temperature ($r_H = 0$), one sees that the radial components of G_3 dualize with an extra Schwarzschild factor $f(r)$ coming from g^{rr}.

To handle this simply we split the forms T_3 and V_3 into radial and angular parts, using

$$
dy^m \equiv y^m d(\ln r) + \omega^m\,, \tag{3.12}
$$

where ω^m is a set of angular 1-forms satisfying

$$
d\omega^m = -dy^m \wedge d(\ln r)\,. \tag{3.13}
$$

Then we have

$$
T_3 = \hat{T}_3 + \tilde{T}_3 \tag{3.14}
$$

with

$$
\begin{aligned}
\hat{T}_3 &= \frac{1}{2} T_{mnp} y^m d(\ln r) \wedge \omega^n \wedge \omega^p\,, \\
&= \frac{1}{2} T_{mnp} y^m d(\ln r) \wedge dy^n \wedge dy^p\,, \\
&= V_3\,, \\
\tilde{T}_3 &= \frac{1}{3!} T_{mnp} \omega^m \wedge \omega^n \wedge \omega^p\,.
\end{aligned} \tag{3.15}
$$

Note that $V_3 = d(\ln r) \wedge S_2 = \hat{V}_3$ has no angular part. The differentials are

$$d\hat{T}_3 = -3d(\ln r) \wedge \tilde{T}_3 \,, \qquad d\tilde{T}_3 = d\hat{V}_3 \,. \tag{3.16}$$

Under flat $*_6$ duality, we have

$$*_6(\hat{T}_3 + \tilde{T}_3) = \pm i(\hat{T}_3 + \tilde{T}_3) \tag{3.17}$$

so that $*_6\hat{T}_3 = \pm i\tilde{T}_3$ and $*_6\tilde{T}_3 = \pm i\hat{T}_3$. Under the duality (3.10) of the 10-dimensional metric (2.1), one thus has

$$\begin{aligned} *\hat{T}_3 &= gf(r) *_6 \hat{T}_3 \wedge \chi_4 = \pm igf(r)\tilde{T}_3 \wedge \chi_4 \,, \\ *\tilde{T}_3 &= g *_6 \tilde{T}_3 \wedge \chi_4 = \pm ig\hat{T}_3 \wedge \chi_4 \,. \end{aligned} \tag{3.18}$$

For V_3 (3.7) implies

$$*V_3 = *\hat{T}_3 = \pm igf(r)\tilde{T}_3 \wedge \chi_4 \,. \tag{3.19}$$

Combining these duality relations with the ansatz (3.5), we find

$$\begin{aligned} *G_3 &= \alpha(r) * (\hat{T}_3 + \tilde{T}_3) + \beta(r) * \hat{V}_3 \\ &= \pm i\Big[\alpha(r)(f(r)\tilde{T}_3 + \hat{T}_3) + \beta(r)f(r)\tilde{T}_3\Big] \wedge \chi_4 \,. \end{aligned} \tag{3.20}$$

Finally we note that

$$G_3 \wedge F_5 = G_3 \wedge (d\chi_4 + *d\chi_4) = -d(G_3 \wedge \chi_4) \,, \tag{3.21}$$

where we have used the fact that $G_3 \wedge *d\chi_4 = 0$ and $dG_3 = 0$.

Thus the equation of motion can be simply rewritten as

$$\begin{aligned} d[*G_3 - iG_3 \wedge \chi_4] &= i\ d\Big(\pm\Big[\alpha(r)(f(r)\tilde{T}_3 + \hat{T}_3) + \beta(r)f(r)\tilde{T}_3 - \\ &\qquad - (\alpha(r)T_3 + \beta(r)\hat{T}_3)\Big] \wedge \chi_4\Big) \\ &= 0 \,. \end{aligned} \tag{3.22}$$

Using

$$d\chi_4 = \left(\frac{Z'}{Z}\right) dr \wedge \chi_4 = -4d(\ln r) \wedge \chi_4 \tag{3.23}$$

and (3.16) it is straightforward to work out the differential, substitute $\beta = r\alpha'/3$, and obtain the differential equation for $\alpha(r)$:

$$rf(r)\alpha''(r) + (11f(r) + rf'(r))\alpha'(r) + 3\left(f'(r) + \frac{7f(r)}{r} + \gamma r\right)\alpha(r) = 0 \,, \tag{3.24}$$

where $\gamma = -7$ in the self dual case $*_6T_3 = iT_3$ and $\gamma = 1$ in the anti-self-dual case $*_6T_3 = -iT_3$. We now restrict to the latter case for which the fluctuation G_3 is in the

$\overline{\mathbf{10}}$ representation dual to the fermion mass term $\operatorname{tr}\lambda\lambda$ in the $\mathcal{N}=1^*$ field theory. Inserting the specific form of $f(r)$ and making the substitutions $\alpha(r)=F(u)/r^4$ with $u=r_H^2/r^2$, one finds the Legendre equation

$$F''(u)-\frac{2u}{1-u^2}F'(u)-\frac{1}{4(1-u^2)}F(u)=0\,. \tag{3.25}$$

The general solution is

$$\alpha(r)=\frac{1}{r^4}\left[aP_{-1/2}\left(\frac{r_H^2}{r^2}\right)+bQ_{-1/2}\left(\frac{r_H^2}{r^2}\right)\right]. \tag{3.26}$$

Only the first term is regular at the horizon, so we set $b=0$ and observe that this solution has the expected [17] behavior $1/r^4$ as $r\to\infty$ associated with the fermion bilinear $\operatorname{tr}\lambda\lambda$ which is an operator of scale dimension 3. The coefficient a is set by comparing (3.26) to the zero temperature result [17], giving the solution

$$\alpha(r)=-\frac{3\sqrt{2}\Gamma(3/4)^2}{g\sqrt{\pi}}\frac{R^4}{r^4}P_{-1/2}\left(\frac{r_H^2}{r^2}\right). \tag{3.27}$$

We also observe that there is no regular solution which approaches the boundary at the rate $1/r^6$ associated with a vacuum expectation value $\langle\operatorname{tr}\lambda\lambda\rangle$.

In the zero temperature case, G_3 is singular at the origin. It was thus necessary to insert D5 or NS5-branes wrapping two-spheres in order to interpret the singularity. In this case, where there is a black hole with a finite Schwarzschild radius, all singularities are hidden behind the horizon. Hence, one is no longer compelled to insert 5-branes. From the gauge theory point of view, one should interpret this as being in the high temperature phase where the system acts like a finite temperature $\mathcal{N}=4$ theory. Only when r_H shrinks to zero will the solution for $\alpha(r)$ be singular, and thus necessitate the insertion of 5-branes.

Let us now turn to the 6-form dual potentials. Making use of the Bianchi identity and the relation of V_3 to T_3, we can rewrite the expression for G_3 in (3.5) as

$$G_3=\alpha(r)\widetilde{T}_3+\left(\alpha(r)+\left(\frac{1}{3}\right)r\alpha'(r)\right)\hat{T}_3\,. \tag{3.28}$$

Equation (3.22) implies that $*G_3-iG_3\wedge\chi_4$ is a closed 7-form. Hence, it can be written as [17]

$$\begin{aligned}*G_3-iG_3\wedge\chi_4 &= \frac{g}{i}d(B_6-\tau C_6)\\ &= \frac{g}{i}d\left(\frac{-i}{3}\frac{\alpha(r)+f(r)(\alpha(r)+r\alpha'(r)/3)}{Z}S_2\right)\times\\ &\quad\times dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,,\end{aligned} \tag{3.29}$$

where the last line is derived using the relations in (3.7). If we assume that $C = 0$, and hence τ is imaginary, then C_6 is given by

$$C_6 = -\frac{g}{3}\left(\alpha(r) + f(r)\left(\alpha(r) + \frac{r\alpha'(r)}{3}\right)\right) \operatorname{Im} S_2 \wedge \chi_4 \,. \tag{3.30}$$

It is instructive to compare this result for C_6 to the NS-NS 2-form. Using (3.2) and (3.9), one finds that

$$B_2 = -\frac{g}{3}\alpha(r) \operatorname{Im} S_2 \,. \tag{3.31}$$

Since $f(r) = 0$ at the horizon, we learn that $C_6 = B_2 \wedge \chi_4$ at $r = r_H$. This confirms the claim in section 2 that a D5-brane with D3-brane charge n has the same action at the horizon as n D3-branes.

4. Back reaction of the perturbation on the dilaton

In this section we compute the order m^2 corrections to the dilaton due to the temperature dependent G_3 perturbation of section 3. The modified dilaton actually contributes only higher order corrections to many of the physical effects we are interested in, but the calculation is simpler than for the 5-form and metric and serves as a prototype. The result is of interest for $T = 0$, since the order m^2 correction to the dilaton, and hence the gauge coupling, is independent of the vacuum state.

We must solve the equation

$$D^M D_M \Phi = -\frac{1}{12} H_{MNP} H^{MNP} + \frac{g^2}{12} \tilde{F}_{MNP} \tilde{F}^{MNP} \,, \tag{4.1}$$

which was obtained from the more general dilaton equation of motion in [17] by specializing dilaton and axion sources to their constant values in (2.2) and (2.3). The metric is that of (2.1) with perpendicular part rewritten in terms of "Cartesian" coordinates y^m as in (3.11). The source can be expressed in terms of G_3 as

$$\begin{aligned} D^M D_M \Phi &= \frac{g^2}{24} g^{mm'} g^{nn'} g^{pp'} \left[G_{mnp} G_{m'n'p'} + \text{c.c.} \right] \\ &= \frac{g^2}{24 Z^{3/2}} \left[G_{mnp} G_{mnp} + \frac{3(f(r)-1)}{r^2} y^m G_{mnp} y^{m'} G_{m'np} + \text{c.c.} \right] \end{aligned} \tag{4.2}$$

in which lower pairs of indices are summed over their 6 values. Using the representation (3.5) and the fact that $y^m V_{mnp} = y^m T_{mnp}$, we obtain

$$\begin{aligned} & G_{mnp} G_{mnp} + \frac{3(f(r)-1)}{r^2} y^m G_{mnp} y^{m'} G_{m'np} = \\ & = \alpha^2(r) T_{pqr} T_{pqr} - \frac{3\alpha^2(r) - 3f(r)(\alpha(r) + r\alpha'(r)/3)^2}{r^2} y^m y^n T_{mpq} T_{npq} \,, \end{aligned} \tag{4.3}$$

where $\alpha(r)$ is the same functions derived in section 3.

We now need the specific form of the anti-self-dual tensor T_{mnp} whose non-vanishing complex components are [17]

$$T_{\bar{p}\bar{q}r} = T_{p\bar{q}\bar{r}} = T_{\bar{p}q\bar{r}} = m\epsilon_{pqr}\,, \tag{4.4}$$

where ϵ_{pqr} is the Levi-Civita symbol in 3 (complex) dimensions. By conversion to complex coordinates and reconversion, we obtain

$$\begin{aligned} T_{pqr}T_{pqr} &= 0\,, \\ y^m y^n T_{mpq}T_{npq} &= 2m^2 r^2 Y\left(\frac{y^i}{r}\right), \\ Y\left(\frac{y^i}{r}\right) &= \frac{(y^1)^2+(y^2)^2+(y^3)^2-(y^4)^2-(y^5)^2-(y^6)^2}{r^2}\,. \end{aligned} \tag{4.5}$$

The function $Y(y^i/r)$ is a harmonic polynomial for the **20'** representation of SO(6) and has "eigenvalue" $-12/r^2$ of the flat 6-dimensional laplacian in the y^i. Thus the final equation for the lowest order dilaton perturbation is

$$D^M D_M \Phi = -\frac{m^2 g^2}{2Z^{3/2}}\left[\alpha^2(r) - f(r)\left(\alpha(r) + \frac{r\alpha'(r)}{3}\right)^2\right] Y\left(\frac{y^i}{r}\right), \tag{4.6}$$

where we have assumed that m^2 is real to prevent a source term for the Ramond-Ramond scalar field.

The next step is to find the specific form of the laplacian. Since the source depends only on r and angles, we can drop t and $\mathbf{x}$ derivatives. The solution must also be proportional to the harmonic $Y(y^i/r)$ so that angular derivatives just give the eigenvalue -12. Thus

$$\begin{aligned} D^M D_M \Phi = D^M D_M \phi(r) Y\left(\frac{y^i}{r}\right) &= \frac{r^2 f(r)}{R^2}\left[\frac{\partial^2}{\partial r^2} + \left(\frac{5}{r} + \frac{f'}{f}\right)\frac{\partial}{\partial r} - \frac{12}{r^2 f(r)}\right]\phi(r) \times \\ &\quad \times Y\left(\frac{y^i}{r}\right). \end{aligned} \tag{4.7}$$

In the limit of zero temperature ($f(r) \to 1$) (4.7) and (4.6) combine to give the radial differential equation

$$\left(\frac{d^2}{dr^2} + \frac{5}{r}\frac{d}{dr} - \frac{12}{r^2}\right)\phi(r) = -\frac{8m^2 R^4}{r^4}\,. \tag{4.8}$$

This equation has homogeneous solutions $\phi(r) \propto r^2, r^{-6}$ which are the power laws of the irregular and regular solutions for a field of scale dimension $\Delta = 6$ in the **20'** representation of SO(6). This is the second Kaluza-Klein excitation of the dilaton/axion [31]. The inhomogeneous solution is

$$\phi(r) = \frac{m^2 R^4}{2r^2}\,. \tag{4.9}$$

Presumably this non-trivial behavior for the dilaton describes the scale dependence of the field theory gauge coupling due to the introduction of mass for the chiral multiplets. However, we do not clearly understand the angular dependence of $\Phi = \phi(r)Y(y^i/r)$ and thus cannot give a more precise interpretation. For a softly broken $\mathcal{N}=4$ to $\mathcal{N}=2$ theory one finds angular dependence for the effective couplings on the coulomb branch [33]. However, for $\mathcal{N}=4$ broken to $\mathcal{N}=1$ there is no coulomb branch, so the Wilsonian interpretation seems less clear.

In this connection it is interesting to note that for the $(T=0)$ solution $\alpha(r) \sim 1/r^6$, which is dual to the field theory vev $\langle \mathrm{tr}(\lambda\lambda)\rangle$, the source term in (4.6) vanishes. Thus the dilaton does not run in a situation where no masses are turned on [33, 34].

In [17] an interpolating function for the dilaton was given that followed a 5-brane geometry in the IR to a 3-brane geometry in the UV. For example, for a vacuum that corresponds to one D5-brane with D3-brane charge N, the interpolating dilaton was given by

$$e^{2\Phi} = g^2 \frac{\rho_-^2}{\rho_-^2 + \rho_c^2}\,, \tag{4.10}$$

where

$$\begin{aligned} \rho_-^2 &= {y_1}^2 + {y_2}^2 + {y_3}^2 + \left(\sqrt{{y_4}^2 + {y_5}^2 + {y_6}^2} - r_0\right)^2, \\ \rho_c &= \frac{2 g r_0 \alpha'}{R^2}\,, \qquad r_0 = \pi\alpha' m N\,. \end{aligned} \tag{4.11}$$

If we expand this for large r we find that the order m^2 dilaton correction is

$$\delta\Phi(y_i) = -\frac{m^2 R^4}{8r^2}\,. \tag{4.12}$$

This is an SO(6) singlet, hence the first order correction in the interpolating dilaton is not quite consistent with the supergravity equations of motion. If one were to have considered an NS5-brane, then one would have found (4.12) but with the opposite sign. In fact, if one were to solve the supergravity equations but with 5-brane sources inserted, one would find that their effects are of higher order in m. The order m^2 term should be independent of the particular vacuum the $\mathcal{N}=1^*$ theory is in.

At finite temperature the equation in (4.6) is more complicated but tractable. Using the variable $z = Z(r) = R^4/r^4$ and setting $k = r_H^4/R^4$, we find the radial differential operator

$$\begin{aligned} \left[\frac{\partial^2}{\partial r^2} + \left(\frac{5}{r} + \frac{f'}{f}\right)\frac{\partial}{\partial r} - \frac{12}{r^2 f(r)}\right]\phi(r) = 16\frac{z^{5/2}}{R^2}\left[\frac{d^2}{dz^2} - \frac{k}{1-kz}\frac{d}{dz} - \frac{3}{4z^2(1-kz)}\right] \times \\ \times\, \phi(z) \end{aligned} \tag{4.13}$$

which is essentially hypergeometric. Its zero modes are:

$$\phi_1(z) = z^{3/2} F\left(\frac{3}{2}, \frac{3}{2}; 3; kz\right) \longrightarrow z^{3/2}\,, \tag{4.14}$$

$$\phi_2(z) = z^{3/2} F\left(\frac{3}{2}, \frac{3}{2}; 1; 1-kz\right) \longrightarrow \frac{4}{\pi k^2 z^{1/2}} \tag{4.15}$$

whose Wronskian is

$$W(\phi_1, \phi_2) = -\frac{8}{\pi k^2 (1-z)}\,. \tag{4.16}$$

The limiting form of the solutions at the boundary ($z \to 0$) is also given in (4.14). At the horizon ($z \to 1/k$), the function $\phi_1(z)$ has a logarithmic singularity while $\phi_2(z)$ is regular.

The finite temperature equation with source (4.6) can be rewritten as

$$\left[\frac{d^2}{dz^2} - \frac{k}{1-kz}\frac{d}{dz} - \frac{3}{4z^2(1-kz)}\right]\phi(r) = \frac{S(z)}{1-kz} \tag{4.17}$$

$$S(z) \equiv -\frac{m^2 g^2 R^2}{32 z^{7/2}} \times \tag{4.18}$$

$$\times \left[\alpha^2(z) - f(z)\left(\alpha(z) - \frac{4\alpha'(z)}{3}\right)^2\right].$$

The method of variation of parameters gives the solution

$$\phi(r) = -\frac{\pi k^2}{8}\left[\phi_1(z)\int_z^{1/k} dz\phi_2(z)S(z) + \phi_2(z)\int_0^z dz\phi_1(z)S(z)\right] \tag{4.19}$$

which is regular on the horizon and vanishes on the boundary at the same rate as (4.9). This solution does not appear to be integrable in closed form.

5. Back reaction to the metric and self dual 5-form

In this section we compute leading order corrections to the metric and self dual 5-form when the field G_3 is turned on. The Einstein equations are

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{g^2}{48}F_{MPQRS}F_N{}^{PQRS} + \frac{g^2}{8}\left(G_{MPQ}G^{*}_{N}{}^{PQ} + c.c\right) -$$

$$-\frac{g^2}{24}g_{MN}G_{PQR}G^{*PQR}\,, \tag{5.1}$$

where the contribution of the dilaton is ignored since it affects only higher order terms.

Let us consider the G_3 terms first. Using the results in (3.2), (3.9) and (3.11), it is straightforward to show that

$$\frac{1}{8}G_{mpq}G_n^{*\,pq} + \text{c.c.} = \frac{|T_3|^2}{24Z}\left(\alpha(r)^2\left(3I_{mn} - 2W_{mn}\right) + \left(\alpha(r) + \frac{r\alpha'(r)}{3}\right)^2 \times \right.$$
$$\left.\times\left(\frac{y^m y^n}{r^2} + 2f(r)(W_{mn} + I_{mn})\right)\right), \tag{5.2}$$

where $\alpha(r)$ is the same as in (3.27), $|T_3|^2 = T_{pqr}T^{*pqr}$ and

$$\begin{aligned} I_{mn} &= \frac{1}{5}\left(\delta_{mn} - \frac{y^m y^n}{r^2}\right), \\ W_{mn} &= \frac{3}{|T_3|^2}\left(T_{mpk}T^*_{npl}\frac{y^k y^l}{r^2} + c.c\right) - I_{mn}\,. \end{aligned} \tag{5.3}$$

Using the fact that

$$\frac{1}{2}(T_{mpq}T^*_{npq} + c.c) = \frac{1}{6}\delta_{mn}|T_3|^2\,, \tag{5.4}$$

one finds that W_{mn} is traceless. We also see that W_{mn} is strictly an angular tensor since $y^m W_{mn} = 0$ because of the antisymmetry of T_{mnp}.

Taking the trace of the expression in (5.2) gives

$$G_{mpq}G^{*mpq} = \frac{|T_3|^2}{2Z^{3/2}}\left(\alpha(r)^2 + \left(\alpha(r) + \frac{r\alpha'(r)}{3}\right)^2\right). \tag{5.5}$$

Hence the contribution to the energy-momentum tensor from the G_3 kinetic terms is

$$\frac{g^2}{8}G_{mpq}G_n^{*\,pq} + \text{c.c.} - \frac{g^2}{24}g_{mn}G_{pqr}G^{*pqr} = \tag{5.6}$$
$$= \frac{g^2|T_3|^2}{48Z}\left(\alpha(r)^2 - f(r)\left(\alpha(r) + \frac{r\alpha'(r)}{3}\right)^2\right)\left(I_{mn} - 4W_{mn} - f^{-1}(r)\frac{y^m y^n}{r^2}\right).$$

for the orthogonal components and

$$\begin{aligned} -\frac{1}{24}g_{ij}G_{pqr}G^{*pqr} &= -\frac{|T_3|^2}{48Z^2}\left(\alpha(r)^2 + \left(\alpha(r) + \frac{r\alpha'(r)}{3}\right)^2\right)\eta_{ij}\,, \\ -\frac{1}{24}g_{00}G_{pqr}G^{*pqr} &= \frac{|T_3|^2}{48Z^2}\left(\alpha(r)^2 + \left(\alpha(r) + \frac{r\alpha'(r)}{3}\right)^2\right)f(r)\,, \end{aligned} \tag{5.7}$$

for the space-time components.

We next compute the corrections to $\widetilde{F}_5 = F_5 - \frac{1}{2}(C_2 \wedge H_3 - B_2 \wedge F_3)$, which in terms of G_3 is

$$\widetilde{F}_5 = F_5 + \frac{ig}{4}(\psi \wedge G_3^* - \psi^* \wedge G_3)\,, \tag{5.8}$$

where $\psi = \frac{1}{3}\alpha(r)S_2$. The equation for motion for $\widetilde{F}_5$ is

$$\widetilde{F}_5 = *\widetilde{F}_5\,. \tag{5.9}$$

There are two correction terms that need to be added to the lowest order term

$$F_5^0 = d\chi_{4,0} + *d\chi_{4,0} \tag{5.10}$$

in order to satisfy (5.9), where $\chi_{4,0}$ is the 4-form field in (2.5). First, since there are corrections to the metric, $\chi_{4,0}$ is no longer harmonic and so the expression in (5.10) is not closed. We will return to this point once we discuss corrections to the metric. Second, the improvement term in (5.8) is not self-dual, so it is necessary to add another correction term to F_5 to compensate for this term. A good guess is to add the dual of the improvement term to F_5, which would make the sum of this term and the improvement term manifestly self-dual. However, one first needs to verify that the dual is closed. To see this, we note that

$$\begin{aligned} G_3 &= \frac{1}{3}\alpha'(r)dr\wedge S_2 + \alpha(r)T_3 \\ &= \left(\frac{1}{3}\alpha'(r)dr + r^{-1}\alpha(r)\right)\wedge S_2 + \frac{1}{3}r^3\alpha(r)d(r^{-3}S_2)\,, \end{aligned} \tag{5.11}$$

where the second term on the r.h.s. of the second line in (5.11) has no radial components. Therefore, we find that

$$\frac{ig}{4}(\psi\wedge G_3^* - \psi^*\wedge G_3) = \frac{ig}{12}\alpha^2(r)\left(S_2\wedge T_3^* - S_2^*\wedge T_3\right), \tag{5.12}$$

with no radial components. Therefore, the dual of (5.12) is

$$\begin{aligned} &\frac{ig\alpha^2(r)}{12Z^2} *_6 (S_2\wedge T_3^* - S_2^*\wedge T_3)\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3 = \\ &\qquad = -\frac{g\alpha(r)^2|T_3|^2}{72Z^2} r dr\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,, \end{aligned} \tag{5.13}$$

which only has a radial component in the orthogonal directions. Since $\alpha(r)$ and Z only have r dependence, the expression in (5.13) is clearly closed.

There is an m^2 correction coming from the cross term of the improvement term and its dual with F_5^0. For the angular, i.e. magnetic, components, the contribution of this cross term to (5.1) is

$$\begin{aligned} &-\frac{ig^2\alpha^2(r)rZ'}{288Z^2}\epsilon^6_{ijkmlq}\frac{y^iy^p}{r^2}\Big(3\left(T_{pjk}T^*_{nlq} - \text{c.c.}\right) - 2\left(T_{pjn}T^*_{klq} - \text{c.c.}\right)\Big) = \\ &\qquad = -\frac{g^2\alpha^2(r)|T_3|^2}{36Z}\left(\delta_{mn} - \frac{y^my^n}{r^2},\right), \end{aligned} \tag{5.14}$$

where we have used the fact that $*_6 T_3 = -iT_3$. It then follows that the electric cross term piece is

$$\frac{g^2\alpha^2(r)|T_3|^2}{36Z^2} g_{MN}\,, \tag{5.15}$$

where g_{MN} is the AdS_5 Schwarzschild metric given in (2.1)

We can now combine the results of (5.6), (5.7), (5.14) and (5.15) into one source term, J_{MN}, for the r.h.s. of (5.1). For the orthogonal components we have

$$\begin{aligned} J_{mn} &= -\frac{g^2|T_3|^2}{144Z}\left(\alpha^2(r) + 3f(r)\left[\alpha(r) + \frac{r\alpha'(r)}{3}\right]^2\right)\left(5I_{mn} - f^{-1}(r)\frac{y^m y^n}{r^2}\right) - \\ &\quad - \frac{g^2|T_3|^2}{12Z}\left(\alpha^2(r) - f(r)\left[\alpha(r) + \frac{r\alpha'(r)}{3}\right]^2\right)\left(W_{mn} + I_{mn}\right), \end{aligned} \tag{5.16}$$

while the space-time components are given by

$$J_{ij} = +\frac{g^2|T_3|^2}{144Z}\left(\alpha^2(r) - 3f(r)\left[\alpha(r) + \frac{r\alpha'(r)}{3}\right]^2\right)\eta_{ij}\,, \tag{5.17}$$

$$J_{00} = \frac{g^2|T_3|^2}{144Z}\left(\alpha^2(r) - 3f(r)\left[\alpha(r) + \frac{r\alpha'(r)}{3}\right]^2\right)f(r)\,. \tag{5.18}$$

Upon inspection of (5.16) one sees that the right-hand side consists of three types of terms. The radial term $y^m y^n/r^2$ and the two angular terms I_{mn} and W_{mn}. Hence, the corrections to the transverse metric components should be of this same type. Likewise, (5.17) indicates that the spatial piece of the metric can have a correction proportional to η_{ij} while (5.18) indicates that the temporal component has a correction proportional to $f(r)$. Hence, we can write down the following ansatz for the metric

$$\begin{aligned} ds^2 = &-\left(Z^{-1/2} + h_0(r)\right) f(r)dt^2 + \left(Z^{-1/2} + h_1(r)\right)\eta_{ij}dx^i dx^j + \\ &+ \Big[\left(5Z^{1/2} + p(r)\right) I_{mn} + f^{-1}(r)\left(Z^{1/2} + q(r)\right)\frac{y^m y^n}{r^2} + \\ &+ w(r)W_{mn}\Big] dy^m dy^n\,. \end{aligned} \tag{5.19}$$

Let us now return to the corrections to F_5 induced by metric corrections. Let us write the 4-form $\chi_4(r) = \chi_{4,0}(r) + \chi_{4,1}(r)$, where $\chi_{4,1}(r)$ is assumed to have components in the four space-time directions,

$$\chi_{4,1}(r) = \chi_1(r)dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\,. \tag{5.20}$$

Therefore, the dual of $d\chi_4(r)$ is

$$\begin{aligned} *\left(d\chi_{4,0}(r) + d\chi_{4,1}(r)\right) &= \frac{1}{5}\epsilon^6_{ijklmn}(\det g_{||})^{-1}\sqrt{-\det g}\,g^{np}\frac{y^p}{r}\left(-\frac{-Z'}{gZ^2} + \chi_1'(r)\right)\times \\ &\quad \times dy^i \wedge dy^j \wedge dy^k \wedge dy^l \wedge dy^m\,. \end{aligned} \tag{5.21}$$

Up to first order corrections this is

$$*(d\chi_0(r) + d\chi_1(r)) = *_0 d\chi_0(r) + \left[-\frac{Z'}{2gZ^{3/2}}\left(\frac{p(r)-q(r)}{Z} - h_0(r) - 3h_1(r)\right) + \right. \\ \left. + \chi_1'(r)\right] *_6 d(\ln r)\,, \tag{5.22}$$

where $*_0$ refers to the dual in the background metric. Hence, $\chi_4(r)$ is harmonic with respect to the new metric if the term in square brackets in (5.22) is zero. Thus, we set

$$\chi_1'(r) = \frac{Z'}{2gZ^{3/2}}\left(\frac{p(r)-q(r)}{Z} - h_0(r) - 3h_1(r)\right), \tag{5.23}$$

Plugging in this value for $\chi_1(r)$, one finds that the leading order corrections to the stress tensor coming from the cross term of $d\chi_{4,1}(r)$ and $d\chi_{4,0}(r)$, denoted by K_{MN}, are

$$K_{mn} = \frac{(Z')^2}{4Z^{5/2}}\left\{-4p(r)I_{mn} + w(r)W_{mn} + [p(r)-q(r)]f^{-1}(r)\frac{y^m y^n}{r^2}\right\} \tag{5.24}$$

for the orthogonal components and

$$\begin{aligned} K_{ij} &= \frac{(Z')^2}{4Z^{5/2}}\left[p(r) - Zh_1(r)\right]\eta_{ij}\,, \\ K_{00} &= -\frac{(Z')^2}{4Z^{5/2}}\left[p(r) - Zh_0(r)\right]f(r) \end{aligned} \tag{5.25}$$

for the space-time components.

The next step is to compute the linear corrections to the Einstein tensor by inserting these expressions for J_{MN} and K_{MN} into (5.1). The space-time components will have the form

$$\begin{aligned} R_{00} - \frac{1}{2}g_{00}R &= -N_0(r)f(r)\,, \\ R_{ij} - \frac{1}{2}g_{ij}R &= N_1(r)\eta_{ij} \end{aligned} \tag{5.26}$$

while the orthogonal components will again be of the form

$$R_{mn} - \frac{1}{2}g_{mn}R = N_2(r)f^{-1}(r)\frac{y^m y^n}{r^2} + N_3(r)I_{mn} + N_4(r)W_{mn}\,. \tag{5.27}$$

Computing the functions in (5.26) and (5.27) is an extremely tedious exercise. We just state the results here. We find that

$$N_0(r) = \frac{1}{Z^{3/2}}\left[\frac{1}{2}f(r)p''(r) + (1+3f(r))\frac{1}{r}p'(r) + (4+3f(r))\frac{1}{r^2}p(r) - \right. \\ - \frac{3}{2}f(r)\frac{1}{r}q'(r) - (6+3f(r))q(r) + \frac{3Z}{2}f(r)h_1''(r) +$$

$$
\begin{aligned}
&+\frac{3Z}{2}(2-f(r))\frac{1}{r}h_1'(r)-6Z\frac{1}{r^2}h_1(r)-4Z\frac{1}{r^2}h_0(r)\Big]\,,\\
N_1(r)=\;&\frac{1}{Z^{3/2}}\Big[\frac{1}{2}f(r)p''(r)+2(1+f(r))\frac{1}{r}p'(r)+(6+f(r))\frac{1}{r^2}p(r)-\\
&-\frac{1}{2}(2+f(r))\frac{1}{r}q'(r)-(8+f(r))q(r)+Zf(r)h_1''(r)+\\
&+Z(4-3f(r))\frac{1}{r}h_1'(r)-Z(12-4f(r))\frac{1}{r^2}h_1(r)+\frac{Z}{2}f(r)h_0''(r)+\\
&+\frac{Z}{2}(6-5f(r))\frac{1}{r}h_0'(r)-Z(6-4f(r))\frac{1}{r^2}h_0(r)\Big]\,,\\
N_2(r)=\;&\frac{1}{Z^{1/2}}\Big[(1+f(r))\frac{1}{r}p'(r)+2(2+f(r))\frac{1}{r^2}p(r)-10q(r)+\\
&+\frac{3Z}{2}(2+f(r))\frac{1}{r}h_1'(r)-Z(6+3f(r))\frac{1}{r^2}h_1(r)+\frac{3Z}{2}f(r)\frac{1}{r}h_0'(r)-\\
&-3Zf(r)\frac{1}{r^2}h_0(r)\Big]\,,\\
N_3(r)=\;&\frac{1}{Z^{1/2}}\Big[2f(r)f_1''(r)+(8+5f(r))\frac{1}{r}f_1'(r)+(26+8f(r))\frac{1}{r^2}f_1(r)-\\
&-5(1+f(r))\frac{1}{r}q'(r)-10(6+f(r))q(r)+\frac{15Z}{2}f(r)h_1''(r)+\\
&+10Z(2-f(r))\frac{1}{r}h_1'(r)-15Z(4-f(r))\frac{1}{r^2}h_1(r)+\frac{5Z}{2}f(r)h_0''(r)+\\
&+5Z(3-2f(r))\frac{1}{r}h_0'(r)-15Z(2-f(r))\frac{1}{r^2}h_0(r)\Big]\,,\\
N_4(r)=\;&\frac{1}{Z^{1/2}}\Big[-\frac{1}{2}f(r)w''(r)-\frac{1}{2}(4+5f(r))\frac{1}{r}w'(r)+(6-2f(r))\frac{1}{r^2}w(r)\Big]\,. \qquad (5.28)
\end{aligned}
$$

Using equations (5.16)–(5.18), (5.24)–(5.25) and (5.28) we can reduce (5.1) to five coupled inhomogenous equations. We write these as

$$
\begin{aligned}
L_0(z)&=\frac{1}{12}A(z)-\frac{1}{4}B(z)\,,\\
L_1(z)&=\frac{1}{12}A(z)-\frac{1}{4}B(z)\,,\\
L_2(z)&=\frac{1}{12}A(z)+\frac{1}{4}B(z)\,,\\
L_3(z)&=-\frac{17}{12}A(z)-\frac{1}{4}B(z)\,,\\
L_4(z)&=-A(z)+B(z)\,, \qquad (5.29)
\end{aligned}
$$

where again $z=R^4/r^4$ and

$$
\begin{aligned}
A(z)&=\frac{g^2|T_3|^2R^2}{12z}\alpha^2(z)\,,\\
B(z)&=\frac{g^2|T_3|^2R^2}{12z}\left(\alpha(z)-\frac{4}{3}z\alpha'(z)\right)^2(1-kz)=\sigma(z)(1-kz)\,, \qquad (5.30)
\end{aligned}
$$

with $k = r_H^4/R^4$. The $L_i(z)$ functions are linear combinations of the metric components and are given by

$$\begin{aligned}
L_0(z) &= 8(1-kz)z^2p''(z) - 2(3-kz)zp'(z) + 3(1-kz)p(z) + 6(1-kz)zq'(z) - \\
&\quad - 3(3-kz)q(z) + 24(1-kz)z^3h_1''(z) + 12(2-3kz)z^2h_1'(z) - \\
&\quad - 6zh_1(z)\,, \\
L_1(z) &= 8(1-kz)z^2p''(z) - 2(3+kz)zp'(z) + (3-kz)p(z) + 2(3-kz)zq'(z) - \\
&\quad - (9-kz)q(z) + 16(1-kz)z^3h_1''(z) + 16(1-2kz)z^2h_1'(z) - \\
&\quad - 4(1+kz)zh_1(z) + 8(1-kz)z^3h_0''(z) + 4(2-5kz)z^2h_0'(z) - \\
&\quad - 2(1+2kz)zh_0(z)\,, \qquad (5.31)\\
L_2(z) &= -4(2-kz)zp'(z) + 2(1-kz)p(z) - 6q(z) - 6(3-kz)z^2h_1'(z) - \\
&\quad - 3(3-kz)zh_1(z) - 6(1-kz)z^2h_0'(z) - 3(1-kz)zh_0(z)\,, \\
L_3(z) &= 32(1-kz)z^2p''(z) - 32zp'(z) + 2(25-4kz)p(z) + 20(2-kz)zq'(z) - \\
&\quad - 10(7-kz)q(z) + 120(1-kz)z^3h_1''(z) + 30(3-7kz)z^2h_1'(z) - \\
&\quad - 15(3+kz)zh_1(z) + 40(1-kz)z^3h_0''(z) + 30(1-3kz)z^2h_0'(z) - \\
&\quad - 15(1+kz)zh_0(z) \\
L_4(z) &= -8(1-kz)z^2w''(z) + 8zw'(z) + 2kzw(z)\,. \qquad (5.32)
\end{aligned}$$

The equations in (5.29) and (5.31) look quite complicated but it turns out that they can be further simplified. First we note that $w(z)$ has decoupled from the other functions and only appears in L_4. To solve for this, write $w(z) = z^{1/2}\widetilde{w}(z)$, where $\widetilde{w}(z)$ is the function in the inertial frame. The last equation in (5.29) can then be rewritten as

$$\widetilde{w}''(z) - \frac{k}{1-kz}\widetilde{w}'(z) - \frac{3}{4z^2}\widetilde{w}(z) = -\frac{A(z)-B(z)}{8z^{1/2}(1-kz)}\,. \qquad (5.33)$$

Using (5.30) we see that this is the same equation as in (4.17), except the source term in (5.33) has an extra factor of -24. Hence the solution for $\widetilde{w}(r)$ is

$$\widetilde{w}(r) = -24\phi(r)\,, \qquad (5.34)$$

where $\phi(r)$ is given in (4.19). The expression in (4.19) is quite complicated, but as we will see, this does not matter since $w(r)$ does not contribute to the lowest order correction to the entropy.

Given (5.33) and the arguments in section 4, it follows that the homogeneous mode for $\widetilde{w}(r)$ corresponds to a dimension 6 operator. Consulting [31, table III], we see that this is the scalar in the **84** representation of SO(6) with $M^2 = 12$. The identity of the two fluctuation equations (4.17) and (5.33), including their sources, suggests that the two fields are in the same supermultiplet, although it is not clear why the sources should agree for T non zero.

Let us next consider the linear combination

$$\frac{5}{8}L_0(z) + \frac{15}{8}L_1(z) + \frac{5}{8}L_2(z) - \frac{3}{8}L_3(z) = \frac{19}{24}A(z) - \frac{3}{8}B(z)\,. \tag{5.35}$$

This reduces to the equation

$$8(1-kz)z^2p''(z) - 8zp'(z) - 2(5+kz)p(z) = \frac{19}{24}A(z) - \frac{3}{8}B(z)\,. \tag{5.36}$$

The solution to this equation with the correct boundary conditions is surprisingly simple,

$$p(z) = -\frac{1}{24}A(z)\,. \tag{5.37}$$

If $T=0$, then the homogeneous solutions for $z^{-1/2}p(z)$ behave like z^2 and z^{-1}. Hence this mode corresponds to a dimension 8 operator. Again consulting [31, table III], this is the SO(6) singlet from $h^a{}_a$ with $M^2 = 32$.

Next, we find that the first two equations and the derivative of the third equation in (5.29) are linearly dependent and that consistent solutions exist provided that the following equation is satisfied:

$$(1-kz)\Big[2zA'(z) + 6zB'(z) - A(z)\Big] - 3(5-3kz)B(z) = 0\,. \tag{5.38}$$

One can easily verify that the expressions for $A(z)$ and $B(z)$ in (5.30) satisfy (5.38).

Of course (5.38) must be true by general covariance; the solutions for $A(z)$ and $B(z)$ came from the equations of motion for G_3. These were derived from the same generally covariant action that the Einstein equations were derived from. The fact that there are only four linearly independent equations means that there is an extra gauge degree of freedom. Thus, we can choose one of the unknown functions to be an arbitrary function. A particularly useful gauge choice is

$$h_1(z) = h_0(z)\,. \tag{5.39}$$

Using (5.39), the third equation in (5.29) reduces to

$$\begin{aligned} -(2-kz)\left(12z^2h_0'(z) + 6zh_0\right) - 6q(z) = (2-kz)\left(-\frac{1}{6}zA'(z) + \frac{1}{12}A(z)\right) + \\ + \frac{1}{4}B(z)\,, \end{aligned} \tag{5.40}$$

where we have substituted the solution in (5.37). Substituting (5.39) into the difference between the second and first equations in (5.29) gives

$$k\left(16z^3h_0'(z)+8z^2h_0(z)-4z^2q'(z)+2zq(z)\right)=k\left(\frac{1}{6}z^2A'(z)-\frac{1}{12}zA(z)\right). \quad (5.41)$$

Examining (5.40) and (5.41), we see that the same linear combination of $h_0(z)$ and $h_0'(z)$ appears, so we can reduce these equations to a single first order equation for $q(z)$. Using (5.38) we can replace all $A(z)$ dependent terms by $B(z)$ and $B'(z)$. The equation can be solved using standard methods, giving the solution

$$q(z)=\frac{1}{24}\left(-B(z)+\frac{2-kz}{\sqrt{z}}\int_0^z\frac{\sigma(z')}{\sqrt{z'}}dz'\right), \quad (5.42)$$

where $\sigma(z)$ is defined in (5.30). An integration constant has been chosen so that $q(z)$ has the correct behavior on the boundary.

To solve for $h_0(z)$ one just substitutes the solution for $q(z)$ in (5.42) into (5.40). Using (5.38) and integrating by parts, one finds

$$h_0(z)=\frac{1}{96z}\left(A(z)-B(z)+\frac{2-kz}{\sqrt{z}}\int_0^z\frac{\sigma(z')}{\sqrt{z'}}dz'\right). \quad (5.43)$$

Clearly, (5.37), (5.42) and (5.43) satisfy

$$p(z)+4zh_0(z)-q(z)=0\,. \quad (5.44)$$

Using these solutions, we can go back and find the corrections to χ_4, the space-time components of C_4. From (5.13) and (5.23), we find that the change in χ_4 is

$$\begin{aligned}\delta\chi_4&=\frac{1}{2g}\int^z\frac{dz}{z^{5/2}}\left(\frac{1}{12}A(z)+\Big[p(z)-q(z)-4zh_0(z)\Big]\right)dx^0\wedge dx^1\wedge dx^2\wedge dx^3\\&=\frac{1}{24g}\int^z\frac{dz'}{z'^{5/2}}\left(B(z')-\frac{2-kz'}{\sqrt{z'}}\int_0^{z'}\frac{\sigma(z'')}{\sqrt{z''}}dz''\right)dx^0\wedge dx^1\wedge dx^2\wedge\\&\qquad\wedge dx^3.\end{aligned} \quad (5.45)$$

The last term can be integrated by parts. Using the relation in (5.30), one is left with

$$\delta\chi_4=\frac{1-kz}{24gz^2}\int_0^z\frac{\sigma(z')}{\sqrt{z'}}dz'dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,. \quad (5.46)$$

In particular, we note that $\delta\chi_4=0$ on the horizon, verifying the claim in section 2 that the r_0^2 term in the probe action goes to zero at the horizon.

A useful check of this analysis is to explicitly compute the m^2 terms that appear in (2.14) at zero temperature. This term comes from corrections to C_4 and the parallel components of the metric. Inserting these corrections into (2.6), one finds that the correction to $-S/V$ is

$$4\pi^2n\mu_5\alpha'\left[\frac{2}{g\sqrt{z}}h_0(z)-\delta C_{0123}(z)\right]. \quad (5.47)$$

Using equations (5.43) and (5.46), the expression in (5.47) reduces to

$$\frac{\pi^2 n\mu_5\alpha'}{12g}\left(A(z)-B(z)+\frac{k}{z}\int_0^z\frac{\sigma(z')}{\sqrt{z'}}dz'\right). \tag{5.48}$$

In the zero temperature limit we set $k=0$. Using (5.30) and the relation

$$|T_3|^2 = 18m^2\,, \tag{5.49}$$

one finds that the correction in (5.48) is

$$\frac{2\pi^2 n\mu_5\alpha'}{g}m^2r^2\,, \tag{5.50}$$

which is precisely the term required by supersymmetry.

We close this section by listing the inhomogeneous solutions for the $T=0$ case. Setting $k=0$ in (4.19), (5.34), (5.37), (5.42) and (5.43), we find for the gauge choice (5.39)

$$\begin{aligned}
w(r) &= -\frac{3m^2R^6}{r^4}\,,\\
p(r) &= -\frac{9m^2R^6}{8r^4}\,,\\
q(r) &= +\frac{m^2R^6}{24r^4}\,,\\
h_0(r) &= +\frac{7m^2R^2}{24}\,.
\end{aligned} \tag{5.51}$$

6. Entropy corrections

In the low temperature phase the masses of the chiral multiplets are much larger than the temperature and therefore the contribution of the chiral multiplets to the entropy is exponentially suppressed. In the high temperature phase, where $T\gg m$, the masses give polynomial corrections to the entropy.

At weak coupling, the entropy density $\mathcal{S}$ is given by

$$\mathcal{S}=\frac{2\pi^2}{3}N^2T^3-\frac{3}{4}N^2m^2T+\mathrm{O}(m^4)\,. \tag{6.1}$$

In [29] it was shown that at strong coupling, the leading order term, $\mathcal{S}_0$, is

$$\mathcal{S}_0=\frac{\pi^2}{2}N^2T^3\,, \tag{6.2}$$

which is 3/4 the value of the leading term in (6.1). In this section we will compute the next term in the series at strong coupling.

To compute the entropy correction, one needs to compute the $\mathrm{O}(m^2)$ correction to the horizon area and the temperature for a fixed Schwarzschild radius. Using the metric in (5.19), we see that the $\mathrm{O}(m^2)$ correction to the area $\mathcal{A}$ for a fixed r_H is

$$\Delta\mathcal{A} = \frac{1}{2} g^{\hat{\mu}\hat{\nu}} \Delta g_{\hat{\mu}\hat{\nu}} \mathcal{A}_0 = \frac{1}{2}\frac{r_H^2}{R^2}\left(3h_1(r_H)\frac{R^4}{r_H^4} + p(r_H)\right)\mathcal{A}_0\,, \tag{6.3}$$

where the $\hat{\mu}$ indices refer to the coordinates orthogonal to the temporal and radial directions. $\mathcal{A}_0$ is the usual $AdS_5 \times S_5$ Schwarzschild area.

Because of the corrections to the metric, it is necessary to adjust the circumference of the euclidean time circle in order to prevent a conical singularity at $r = r_H$. The corresponding correction to the temperature, for fixed r_H, is

$$\Delta T = \frac{1}{2}\frac{r_H^2}{R^2}\left(h_0(r_H)\frac{R^4}{r_H^4} - q(r_H)\right)T\,. \tag{6.4}$$

Hence, using (5.44) and (6.2), we find that the $\mathrm{O}(m^2)$ correction to the entropy for fixed *temperature* is

$$\begin{aligned}\Delta\mathcal{S} &= \left(\frac{\Delta\mathcal{A}}{\mathcal{A}_0} - 3\frac{\Delta T}{T}\right)\mathcal{S}_0\,,\\ &= 2\frac{r_H^2}{R^2}\left(3h_0(r_H)\frac{R^4}{r_H^4} + p(r_H)\right)\mathcal{S}_0\,.\end{aligned} \tag{6.5}$$

If we now use the relations in (5.30), (5.37), (5.43) and (5.49), we find that

$$\Delta\mathcal{S} = -\frac{9}{16}\frac{\Gamma(3/4)^4}{\pi^3}\frac{m^2}{T^2}\left(1 - \frac{2}{3}\int_0^1 \frac{x^2\left[P_{-1/2}(x) - xP_{1/2}(x)\right]^2}{(1-x^2)^2}dx\right)\mathcal{S}_0\,. \tag{6.6}$$

We are unaware of any analytic expression for the above integral, but the numerical result for $\Delta\mathcal{S}$ is

$$\Delta\mathcal{S} = -0.1714 N^2 m^2 T\,, \tag{6.7}$$

which is 23% of the weak coupling result.

This demonstrates the shrinking of the horizon area as the mass is increased. It also suggests that a critical temperature is reached when $T \sim m$. It is hoped that one can carry out a Hawking-Page analysis of this transition [3, 35], by demonstrating that the free energy is lowered if the black-hole geometry of the bulk changes to the Polchinski-Strassler geometry. However, since the full 10-dimensional space is not a product space, in order to proceed one will require a more sophisticated approach for evaluating boundary terms in the euclidean action.

As was case for the T^3 term, the m^2T term in the entropy is somewhat suppressed at strong coupling. In fact as a percentage, the suppression is more. An interesting test would be to carry out the analysis in [36] for the $\mathcal{N} = 1^*$ system. In [36], the first order correction in $1/(gN)$ was computed, and it was shown that the entropy increases when moving away from the strong coupling limit. For the $\mathcal{N} = 1^*$ system, one should find that the absolute value of the m^2T term also increases when moving away from strong coupling.

7. Discussion

The results presented for the backreaction required a lot of tedious calculation. But in the end, the results had significant simplification. This leaves us optimistic that an exact supergravity solution can be found. Recent progress in finding exact solutions for other supergravity duals [33], [37]–[43] should give one hope in succeeding in this case as well.

It might be the case that finding an exact solution for the high temperature phase will be an easier task then finding the solution for the the low temperature phase. While the low temperature phase has unbroken $\mathcal{N} = 1$ supersymmetry, it does have the extra complication of 5-branes wrapping 2-spheres. In any case, armed with an exact solution of the high temperature phase, one should be able to observe the shrinking of the black hole horizon down to zero area for some non-zero $T = T_{\rm c}$. Work on this problem continues.

Acknowledgments

We thank L. Rastelli, M. Strassler and N. Warner for helpful discussions. This research was supported in part by the NSF under grant number PHY-97-22072 and by the U.S. Department of Energy under contract number DE-FC02-94ER40818.

A. Disappearance of metastable vacua

To get a better estimate for the temperature at which a metastable state disappears, we need to compute a discriminant. The derivative of (2.14) is proportional to

$$4Ar^3\left(\sqrt{1-\frac{r_H^4}{r^4}}-1\right)+\frac{2Ar_H^4}{r\sqrt{1-r_H^4/r^4}}+2\left(r^2-rr_0\right)\left(2r-r_0\right), \qquad \text{(A.1)}$$

where $A = (\pi n^2)/(2gN) \gg 1$. Hence, to find extrema for the potential one looks for zeros of the polynomial

$$\begin{aligned}(8A-4)r^8+(12-12A)r^7+(-13+4A)r^6+6r^5+\left(-1-8Ar_H^4+4r_H^4\right)r^4+\\ +\left(-12r_H^4+12Ar_H^4\right)r^3+\left(13r_H^4-4Ar_H^4\right)r^2-6r_H^4r+r_H^4+A^2r_H^8\,. \qquad \text{(A.2)}\end{aligned}$$

In fact, we want to find where two extrema coalesce, which occurs when the discriminant of this polynomial has a zero. The discriminant factors into a seventh order polynomial and and a second order polynomial in r_H^4, and it turns out that it is the zeros of the seventh order polynomial that we need. However, for large A it turns out that this polynomial is approximately

$$2^{29}A^9(r_H^4)^7-3^3(3351797)A^8r_0^8(r_H^4)^5-2^83^6r_0^{16}(r_H^4)^3\,. \qquad \text{(A.3)}$$

Hence, there is a zero at

$$r_H \approx \frac{1}{16} 3^{3/8} 2^{1/4} \left(3351797 + (171)(2659)\sqrt{57}\right)^{1/8} A^{-1/8} r_0 \,, \tag{A.4}$$

and so the solution disappears at

$$r_H \approx \kappa \left(\frac{n^2}{4\pi g N}\right)^{-1/8} r_0 \,, \tag{A.5}$$

where

$$\kappa = \frac{1}{16} \left(\frac{2}{\pi^2}\right)^{1/8} 3^{3/8} \left(3351797 + (171)(2659)\sqrt{57}\right)^{1/8} \approx 0.5522 \,. \tag{A.6}$$

In terms of the temperature and m, this is

$$T = \kappa \left(\frac{n^2}{4\pi g N}\right)^{3/8} |m| \,. \tag{A.7}$$

References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [hep-th/9905111].

[3] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[4] J.G. Russo, *New compactifications of supergravities and large-N QCD*, *Nucl. Phys.* **B 543** (1999) 183 [hep-th/9808117].

[5] C. Csáki, Y. Oz, J. Russo and J. Terning, *Large-N QCD from rotating branes*, *Phys. Rev.* **D 59** (1999) 065012 [hep-th/9810186].

[6] J.G. Russo and K. Sfetsos, *Rotating D3 branes and QCD in three dimensions*, *Adv. Theor. Math. Phys.* **3** (1999) 131 [hep-th/9901056].

[7] M. Cvetič and S.S. Gubser, *Phases of R-charged black holes, spinning branes and strongly coupled gauge theories*, *J. High Energy Phys.* **04** (1999) 024 [hep-th/9902195].

[8] M. Cvetič and S.S. Gubser, *Thermodynamic stability and phases of general spinning branes*, *J. High Energy Phys.* **07** (1999) 010 [hep-th/9903132].

[9] I.R. Klebanov and A.A. Tseytlin, *D-branes and dual gauge theories in type-0 strings*, *Nucl. Phys.* **B 546** (1999) 155 [hep-th/9811035].

[10] J.A. Minahan, *Glueball mass spectra and other issues for supergravity duals of QCD models*, *J. High Energy Phys.* **01** (1999) 020 [hep-th/9811156].

[11] I.R. Klebanov and A.A. Tseytlin, *Asymptotic freedom and infrared behavior in the type-0 string approach to gauge theory*, *Nucl. Phys.* **B 547** (1999) 143 [hep-th/9812089].

[12] J.A. Minahan, *Asymptotic freedom and confinement from type-0 string theory*, *J. High Energy Phys.* **04** (1999) 007 [hep-th/9902074].

[13] A. Kehagias and K. Sfetsos, *On running couplings in gauge theories from type-IIB supergravity*, *Phys. Lett.* **B 454** (1999) 270 [hep-th/9902125].

[14] S.S. Gubser, *Dilaton-driven confinement*, hep-th/9902155.

[15] A. Kehagias and K. Sfetsos, *On asymptotic freedom and confinement from type-IIB supergravity*, *Phys. Lett.* **B 456** (1999) 22 [hep-th/9903109].

[16] S.S. Gubser, *Curvature singularities: the good, the bad and the naked*, hep-th/0002160.

[17] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[18] I. Bena, *The M-theory dual of a 3 dimensional theory with reduced supersymmetry*, *Phys. Rev.* **D 62** (2000) 126006 [hep-th/0004142].

[19] I. Bena and A. Nudelman, *Warping and vacua of (S)YM(2+1)*, *Phys. Rev.* **D 62** (2000) 086008 [hep-th/0005163].

[20] I. Bena and A. Nudelman, *Exotic polarizations of D2 branes and oblique vacua of (S)YM(2+1)*, *Phys. Rev.* **D 62** (2000) 126007 [hep-th/0006102].

[21] C. Bachas, J. Hoppe and B. Pioline, *Nahm equations, $N = 1^*$ domain walls and d strings in $AdS_5 \times S^5$*, hep-th/0007067.

[22] F. Zamora, *Non-supersymmetric* SO(3)*-invariant deformations of $N = 1^*$ vacua and their dual string theory description*, *J. High Energy Phys.* **12** (2000) 021 [hep-th/0007082].

[23] A. Frey, *Brane configurations of BPS domain walls for the $N = 1^*$* SU(N) *gauge theory*, *J. High Energy Phys.* **12** (2000) 020 [hep-th/0007125].

[24] C. Vafa and E. Witten, *A strong coupling test of S-duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074].

[25] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys.* **B 460** (1996) 299 [hep-th/9510101].

[26] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[27] D. Kabat and I.W. Taylor, *Spherical membranes in matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 181 [hep-th/9711078].

JHEP01(2001)036

[28] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *The supergravity dual of $N = 1$ super Yang-Mills theory*, *Nucl. Phys.* **B 569** (2000) 451 [hep-th/9909047].

[29] S.S. Gubser, I.R. Klebanov and A.W. Peet, *Entropy and temperature of black 3-branes*, *Phys. Rev.* **D 54** (1996) 3915 [hep-th/9602135].

[30] G.T. Horowitz and A. Strominger, *Black strings and p-branes*, *Nucl. Phys.* **B 360** (1991) 197.

[31] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *The mass spectrum of chiral $N = 2$ $D = 10$ supergravity on S^5*, *Phys. Rev.* **D 32** (1985) 389.

[32] A.A. Tseytlin and S. Yankielowicz, *Free energy of $N = 4$ super Yang-Mills in Higgs phase and non-extremal D3-brane interactions*, *Nucl. Phys.* **B 541** (1999) 145 [hep-th/9809032].

[33] K. Pilch and N.P. Warner, *$N = 2$ supersymmetric RG flows and the IIB dilaton*, *Nucl. Phys.* **B 594** (2001) 209 [hep-th/0004063].

[34] M. Petrini and A. Zaffaroni, *The holographic RG flow to conformal and non-conformal theory*, hep-th/0002172.

[35] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in anti-de Sitter space*, *Comm. Math. Phys.* **87** (1983) 577.

[36] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *Coupling constant dependence in the thermodynamics of $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 534** (1998) 202 [hep-th/9805156].

[37] K. Pilch and N.P. Warner, *A new supersymmetric compactification of chiral IIB supergravity*, *Phys. Lett.* **B 487** (2000) 22 [hep-th/0002192].

[38] A. Brandhuber and K. Sfetsos, *An $N = 2$ gauge theory and its supergravity dual*, *Phys. Lett.* **B 488** (2000) 373 [hep-th/0004148].

[39] K. Pilch and N.P. Warner, *$N = 1$ supersymmetric renormalization group flows from IIB supergravity*, hep-th/0006066.

[40] I.R. Klebanov and N.A. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, *Nucl. Phys.* **B 574** (2000) 263 [hep-th/9911096].

[41] I.R. Klebanov and A.A. Tseytlin, *Gravity duals of supersymmetric* $\mathrm{SU}(N)\times\mathrm{SU}(N+M)$ *gauge theories*, *Nucl. Phys.* **B 578** (2000) 123 [hep-th/0002159].

[42] K. Oh and R. Tatar, *Renormalization group flows on D3 branes at an orbifolded conifold*, *J. High Energy Phys.* **05** (2000) 030 [hep-th/0003183].

[43] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and χSB-resolution of naked singularities*, *J. High Energy Phys.* **08** (2000) 052 [hep-th/0007191].